Indian Statistical Institute, Bangalore Centre
Solution set of M. Math Mid-Sem Examination 2010
Functional Analysis

1. Let $X$ be a complex normed linear space. Let $f : X \to \mathbb{C}$ be a non-zero linear map. Show that either $f(B(0,1))$ is a bounded set or all of $\mathbb{C}$. In the second case show that $\ker f$ is dense in $X$.

   *Proof.* If $f$ is continuous, then $|f(x)| \leq \|f\|\|x\|$ for all $x \in X$. For $x \in B(0,1)$, i.e., $\|x\| < 1$, we have $|f(x)| \leq \|f\|\|x\| < \|f\|$ for $x \in B(0,1)$. Hence $f(B(0,1))$ is a bounded set.

   Now we show that if $f$ is not continuous, then $f(B(0,1)) = \mathbb{C}$. Since $f : X \to \mathbb{C}$ is not continuous, $f$ is not bounded for $B(0; \frac{1}{n}) = \{x \in X : \|x\| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. So there exists $x_n \in B(0; \frac{1}{n})$ such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$. Take any $\alpha \in \mathbb{C}$. There is some natural number $k$ such that $|\alpha| < k$. Since $|f(x_k)| > k, |\frac{\alpha}{f(x_k)}| < 1$, so $\frac{\alpha}{f(x_k)}x_k$ is inside open unit ball $B(0,1)$. But $f(\frac{\alpha}{f(x_k)}x_k) = \frac{\alpha}{f(x_k)}f(x_k) = \alpha$, so $\alpha$ is in the image of the open unit ball under $f$.

   Our next claim is to show that if $f(B(0,1)) = \mathbb{C}$, then $\ker f$ is dense in $X$. Clearly $f(B(0,1)) = \mathbb{C}$ implies $f$ is not continuous. Since $f : X \to \mathbb{C}$ is not continuous, $f$ is not bounded for $B(0; \frac{1}{n}) = \{x \in X : \|x\| < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. So there exists $x_n \in B(0; \frac{1}{n})$ such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$. Note that $x_n \to 0$ and $|f(x_n)| \to \infty$ as $n \to \infty$. Let $x \in X$. Define $y_n = x - \dfrac{f(x)}{f(x_n)}x_n$ for each $n \in \mathbb{N}$. Then using the linearity of $f$ we can see that $y_n \in \ker f$ for each $n \in \mathbb{N}$. Since $x_n \to 0$ and $|f(x_n)| \to \infty$ as $n \to \infty$, $y_n \to x$ as $n \to \infty$. Therefore $x \in \overline{\ker f}$ and which implies $\overline{\ker f} = X$. Hence $\ker f$ is a dense subspace of $X$. $\square$

2. Show that for any normed linear space $X$, $X^*$ is a Banach space.

   *Proof.* First we prove that if $Y$ is a Banach space, then $BL(X,Y)$ is a Banach space. Let $\{F_n\}$ be a Cauchy sequence in $BL(X,Y)$. For given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\|F_n(x - F_m(x)\| \leq \|F_n - F_m\|\|x\| < \epsilon\|x\| \tag{1}$$

for all $x \in X$ and all $n, m \geq N$. Thus $\{F_n(x)\}$ is a Cauchy sequence in $Y$ for each $x \in X$. Since $Y$ is Banach space, $\{F_n(x)\}$ converges in $Y$, namely to $y_x$. Define a map $F : X \to Y$ by

$$F(x) = y_x = \lim_{n\to\infty} F_n(x)$$

for $x \in X$. Clearly $F$ is a linear map. Since $\{F_n\}$ is a Cauchy sequence, $\{F_n\}$ is a bounded sequence. Assume that $\|F_n\| \leq M$ for some $M > 0$, for all $n$. Then $\|F(x)\| \leq Mx$ for all $x \in X$. Thus $F \in BL(X, Y)$. Now taking $m \to \infty$ and supremum over all $x \in X$ with $\|x\| \leq 1$ in equation (1), we have

$$\|F_n - F\| \leq \epsilon$$

for all $n \geq N$. Thus $F_n \to F$ in $BL(X, Y)$. Hence $BL(X, Y)$ is a Banach space. In particular, for $Y = \mathbb{C}$ we have $B(X, Y) = B(X, \mathbb{C}) = X^*$ = dual of $X$ is a Banach space. □

3. Let $M = \{f \in C[0, 1] : f|_{[0,\frac{1}{2}]} = 0\}$. Let $\Phi : C[0, 1]/M \to C[0, \frac{1}{2}]$ be defined by $\Phi(f + M) = f|_{[0,\frac{1}{2}]}$. Show that $\Phi$ is a well-defined, linear, onto, isometry.

*Proof.* Let us consider $X = C[0, 1]$ and $Y = C[0, \frac{1}{2}]$.

Well-defined/Linear: Let $f, g \in X$ be such that $f + M = g + M$. Then $f - g \in M$. Thus $(f - g)|_{[0,\frac{1}{2}]} = 0$. Hence $f|_{[0,\frac{1}{2}]} = g|_{[0,\frac{1}{2}]}$. So $\Phi$ is well-defined. Clearly $\Phi$ is linear.

Onto: Let $g \in Y$. If we define

$$f(t) := \begin{cases} g(t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ g(1-t) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

then $f \in X$ and $\Phi(f + M) = f|_{[0,\frac{1}{2}]} = g$. Hence $\Phi$ is onto.

Isometry: We recall the definition of the quotient norm $\|f + M\| = \inf\{\|f - g\|_\infty : g \in M\}$. Let $f \in X$. Since $g|_{[0,\frac{1}{2}]} = 0$ for any $g \in M$, we have $\|f - g\|_\infty \geq \|f|_{[0,\frac{1}{2}]}\|_\infty$. Thus $\|f + M\| = \inf\{\|f - g\|_\infty : g \in M\} \geq \|f|_{[0,\frac{1}{2}]}\|_\infty$. Define

$$g_0(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}; \\ f(t) - f(\frac{1}{2}) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

Then $g_0 \in M$. Note that $\|f - g_0\|_\infty = \sup\{|f(t)| : t \in [0, \frac{1}{2}]\} = \|f|_{[0,\frac{1}{2}]}\|_\infty$. Since $g_0 \in M$, $\|f + M\| \leq \|f - g_0\|_\infty = \|f|_{[0,\frac{1}{2}]}\|_\infty$. Thus $\|f + M\| = \|f|_{[0,\frac{1}{2}]}\|_\infty$. Hence $\Phi$ is an isometry. □

4. Let $X$ be a normed linear space and $M$ a proper closed subspace. Let $\pi : X \to X/M$ be the quotient map. Show that $\|\pi\| = 1$.

*Proof.* Observe that $\|\pi(x)\| = \|x+M\| \leq \|x\|$ for all $x \in X$. So $\|\pi\| \leq 1$. For reverse inequality we use the F. Riesz's Lemma:

Let $M$ be a proper, closed subspace of a normed space $X$. Then for given $\epsilon > 0$,there exists $x \in X$ with $\|x\| = 1$, such that

$$\|x + M\| > 1 - \epsilon.$$

So for each $n \in \mathbb{N}$, there exists $x_n \in X$ with $\|x_n\| = 1$, such that

$$\|x_n + M\| > 1 - \frac{1}{n}.$$

Therefore $\|\pi(x_n)\| > 1 - \frac{1}{n}$. Since $\|x_n\| = 1$ for each $n \in \mathbb{N}$, we have $\|\pi\| \geq 1$. Hence $\|\pi\| = 1$. □

5. Let $H$ be a complex separable Hilbert space. Show that for some discrete set $\Delta$, there is a linear, continuous, onto map from $H \to \ell^2(\Delta)$.

*Proof.* Since $H$ is a complex separable Hilbert space, $H$ has a countable orthonormal basis say $\{u_1, u_2, \ldots\}$. For $x \in H$,

$$F(x) = (\langle x, u_1\rangle, \langle x, u_2\rangle, \ldots)$$

If $\{u_1, u_2, \ldots\}$ is a finite set having $n$ elements, then $F$ is a linear map from H to $\mathbb{C}^n = \ell^2(\{1, 2, \ldots, n\})$. If $\{u_1, u_2, \ldots\}$ is a countable infinite set, then Bessel's inequality shows that $F$ is a linear map from $H$ to $\ell^2(\mathbb{N})$. If we consider the norm $\|.\|_2$ on $\ell^2(\mathbb{N})$, then the Parseval formula (for each $x \in H$ we have $\|x\|^2 = \sum_n |\langle x, u_n\rangle|^2$) shows that

$$\|x\|^2 = \sum_n |\langle x, u_n\rangle|^2 = \|F(x)\|_2^2.$$

for each $x \in H$. Hence $F$ is an isometry. Let $y \in \ell^2(\mathbb{N})$. Then $y = \sum_n k_n e_n = (k_1, k_2, \ldots)$ for $k_n \in \mathbb{C}$ where $\{e_n : n = 1, 2, \ldots\}$ is the standard orthonormal basis of $\ell^2(\mathbb{N})$. Note that $\|y\|_2^2 = \sum_n |k_n|^2 < \infty$. By Riesz-Fischer Theorem, $\sum_n k_n u_n$ converges in $H$, say, it converges to $x \in H$, i.e., $x = \sum_n k_n u_n$. Since $\{u_1, u_2, \ldots\}$ is an orthonormal basis of $H$, $\langle x, u_m \rangle = k_m$. Then $F(x) = y$. Hence $F$ is a surjective map. □

6. Let $H$ be a complex Hilbert space. Let $P : H \to H$ be a linear map such that

   1. $P \neq 0$ and $P \neq I$,
   2. $P(P(x)) = P(x)$ for all $x$,
   3. $\|P(x)\|^2 + \|x - P(x)\|^2 = \|x\|^2$ for all $x$.

   Show that $\|P\| = 1 = \|I - P\|$, where $I$ denotes the identity map.

   *Proof.* Note that $P^2 = P \Rightarrow \|P\| = \|P^2\| \leq \|P\|\|P\| \Rightarrow 1 \leq \|P\|$. Since $\|P(x)\|^2 + \|x - P(x)\|^2 = \|x\|^2$ for all $x$, $\|P(x)\|^2 \leq \|x\|^2 \Rightarrow \|P\| \leq 1$. So $\|P\| = 1$.

   Observe that $(I - P)^2 = I - P - P + P^2 = I - P$. In the above arguments, if we change the role of $P$ by $I - P$, then $\|I - P\| = 1$. □

7. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\{f_n\}_{n\geq 1} \subset L^3(P)$ be a sequence such that $f_n \to f$ for some $f \in L^3(P)$. Show that for any $g \in L^{\frac{3}{2}}(P)$, $\int f_n g \; dP \to \int fg \; dP$.

   *Proof.* Let us consider $p = 3$ and $q = \frac{3}{2}$. Then $\frac{1}{p} + \frac{1}{q} = 1$. By using the Holder's inequality, for any $g \in L^{\frac{3}{2}}(P)$ we have

   $$|\int (f_n - f)g \; dP| \leq \int |(f_n - f)g| \; dP \leq \Big(\int |f_n - f|^p \; dP\Big)^{\frac{1}{p}} \Big(\int |g|^q \; dP\Big)^{\frac{1}{q}} \to 0$$

   as $n \to \infty$. Hence for any $g \in L^{\frac{3}{2}}(P)$, $\int f_n g \; dP \to \int fg \; dP$ as $n \to \infty$. □

8. Let $(X, \mathcal{T})$ be a locally compact non-compact space. Show that $C_0(X)$ is a Banach space with the supremum norm.

*Proof.* If we can show that $C_0(X)$ is a closed subspace of the complete space $(C(X), \|.\|_\infty)$, then $C_0(X)$ is also a Banach space with the supremum norm. Let $f$ be any element in the closure of $C_0(X)$ in $C(X)$. Then there is a sequence $\{f_n\}$ in $C_0(X)$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. So for given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \epsilon$ for all $n \geq N$. Since $f_N \in C_0(X)$, there is a compact subset $K \subset X$ such that $|f_N(t)| < \epsilon$ for all $t \notin K$. Hence

$$|f(t)| \leq |f(t) - f_N(t)| + |f_N(t)| \leq \|f_N - f\|_\infty + |f_N(t)| < \epsilon + \epsilon$$

for all $t \notin K$. This shows that $f \in C_0(X)$. Thus $C_0(X)$ is a closed subspace of $C(X)$. So it is complete. □